Enhanced synchronizability by structural perturbations

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In this Brief Report, we investigate the collective synchronization of a system of coupled oscillators on a Barabási-Albert scale-free network. We propose an approach of structural perturbations aiming at those nodes with maximal betweenness. This method can markedly enhance the network synchronizability, and is easy to realize. The simulation results show that the eigenratio will sharply decrease by one-half when only 0.6% of those hub nodes occur under three-division processes when the network size $N=2000$. In addition, the present study also provides numerical evidence that the maximal betweenness plays a major role in network synchronization.

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I. INTRODUCTION

Many social, biological, and communication systems can be properly described as complex networks with nodes representing individuals or organizations and edges mimicking the interactions among them [1]. One of the ultimate goals of the current studies on the topological structure of networks is to understand and explain the workings of systems built upon those networks: for instance, to understand how the topology of the Internet affects the spread of a computer virus [2], how the structure of power grids affects cascading behavior [3], how the connecting pattern of intercommunication networks affects traffic dynamics [4–6], and so on.

In the past few years, with the computerization of the data acquisition process and the availability of high computing power, scientists have found that many real-life networks share some common statistical characteristics, the most important of which are called the small-world effect [7,8] and scale-free property [9]. The recognition of the small-world effect involves two facts, a small average distance [10] (varying as $L\sim \ln N$, where N is the number of nodes in the network) and a large clustering coefficient [11]. The number of edges incident from a node x is called the degree of x. The scale-free property means the degree distribution of the network obeys a power-law form, that is, $p(k)\sim k^{-\gamma}$, where k is the degree and $p(k)$ is the probability density function for the degree distribution. γ is called the power-law exponent, and is usually between 2 and 3 in the real world [1]. This power-law distribution falls off much more gradually than an exponential one, allowing for a few nodes of very large degree to exist. Networks with power-law degree distribution are referred to as scale-free networks, although one can and usually does have scales present in other network properties. The pioneer model of scale-free networks is the Barabási-Albert (BA) model, which suggests that two main ingredients of self-organization of a network in a scale-free structure are growth and preferential attachment [9]. These point to the facts that most networks grow continuously by adding new nodes, which are preferentially attached to existing nodes with a large number of neighbors.

Synchronization is observed in a variety of natural, social, physical, and biological systems [12]. To understand how the network structure affects the synchronizability of the network is of not only theoretical interest, but also practical value. There are many previous studies about collective synchronization, with a basic assumption that the dynamic system of coupled oscillators evolves either on regular networks [13], or on random ones [14]. Very recently, scientists have focused on synchronization on complex networks, and found that the networks with the small-world effect and scale-free property may be easier to synchronize than regular lattices [15–20].

Since there are countless topological characters for networks, a natural question is addressed: What is the most important factor by which the synchronizability of the networks is mainly determined? Some previous works indicated that the average distance L is one of the key factors; a smaller L will lead to better synchronizability [16,17,19]. Other researchers focus on the role played by degree of heterogeneity. They found that that greater heterogeneity will result in poorer synchronizability, and demonstrated that the maximal betweenness [21,22] B_{max} may be a proper quantity to estimate the network synchronizability. With smaller B_{max}, the network synchronizability will be better [23,24]. However, the above results and conclusions are still debated.

In this Brief Report, we investigate the collective synchronization of system of coupled oscillators on Barabási-Albert scale-free networks (BA networks) [9]. We propose an approach of structural perturbations, which can markedly enhance the network synchronizability, and is easy to realize. It also provides numerical evidence that the maximal betweenness plays a main role in network synchronization.

This paper is organized as follows. In Sec. II, the concept of synchronizability will be briefly introduced. In Sec. III, we will describe the approach of structural perturbations. Next, the simulation results will be given. Finally, in Sec. V, the conclusion is drawn, and the relevance of this approach to some real-life problems is discussed.

II. SYNCHRONIZABILITY

We introduce a generic model of coupled oscillators on networks and the master stability function [25], which is of-

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ten used to test the stability of complete synchronized states.

Each node of a network is located an oscillator; a link connecting two nodes represents coupling between the two oscillators. The state of the ith oscillator is described by $\mathbf{x}^i$. We get the set of equations of motion governing the dynamics of the N coupled oscillators:

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) + \sigma \sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}^j), \quad (1)$$

where $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i)$ governs the dynamics of the individual oscillator, $\mathbf{H}(\mathbf{x}^j)$ is the output function, and σ is the coupling strength. The $N \times N$ coupling matrix $\mathbf{G}$ is given by

$$G_{ij} = \begin{cases} -k_i & \text{for } i = j, \\ 1 & \text{for } j \in \Lambda_i, \\ 0 & \text{otherwise}. \end{cases} \quad (2)$$

All the eigenvalues of matrix $\mathbf{G}$ are nonpositive real values because $\mathbf{G}$ is negative semidefinite, and the biggest eigenvalue γ_0 is always zero because the rows of $\mathbf{G}$ have zero sum. Thus, the eigenvalues can be ranked as $\gamma_0 \geqslant \gamma_1 \geqslant \cdots \geqslant \gamma_{N-1}$, and the synchronization manifold is an invariant manifold, that is, the fully synchronized state $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N = \mathbf{s}$ satisfies $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$. It is worthwhile to emphasize that $\gamma_0 = \gamma_1 = 0$ if and only if the network is disconnected.

Let ξ^i be the variation on the ith node, and the collection of variation be $\xi = (\xi^1, \xi^2, \ldots = \xi^N)$. We get the variational equation of (1),

$$\dot{\xi} = (\mathbf{1}_N \otimes D\mathbf{F} + \sigma \mathbf{G} \otimes D\mathbf{H})\xi, \quad (3)$$

where $\otimes$ is the direct product. Diagonalizing $\mathbf{G}$ in the second term of Eq. (3), a block diagonalized variational equation is obtained and each block has the form

$$\dot{\xi}_k = (D\mathbf{F} + \sigma\gamma_k D\mathbf{H})\xi_k, \quad (4)$$

where D denotes the Jacobian matrix, γ_k is an eigenvalue of $\mathbf{G}$, and $k = 0, 1, 2, \ldots, N-1$. $k = 0$ corresponds to the mode that is parallel to the synchronization manifold. Let $\alpha = \sigma\gamma_k$, and rewrite Eq. (4) as

$$\dot{\zeta} = (D\mathbf{F} + \alpha D\mathbf{H})\zeta. \quad (5)$$

Since $D\mathbf{F}$ and $D\mathbf{H}$ are the same for each block, the largest Lyapunov exponent λ_{max} of Eq. (5) only depends on α. The function $\lambda_{max}(\alpha)$ is named the master stability function, whose sign indicates the stability of the mode: the synchronized state is stable if $\lambda_{max}(\alpha) < 0$ for all blocks.

For many dynamical systems, the master stability function is negative in a single finite interval (α_1, α_2) and the largest Lyapunov exponent is negative [26]. Therefore, the network is synchronizable for some σ when the eigenratio $r = \gamma_{N-1}/\gamma_1$ satisfies

$$r < \alpha_2/\alpha_1. \quad (6)$$

The right-hand side of this equation depends only on the dynamics of individual oscillators and the output function, while the eigenratio r depends only on the coupling matrix $\mathbf{G}$. The problem of synchronization is then divided into two parts: choosing suitable parameters of the dynamics to broaden the interval (α_1, α_2) and the analysis of the eigenratio of the coupling matrix. The eigenratio r of the coupling matrix indicates the synchronizability of the network; the smaller it is, the better the synchronizability, and vice versa. In this paper, for universality, we will not address a particular dynamical system, but concentrate on how the network topology affects eigenratio r.

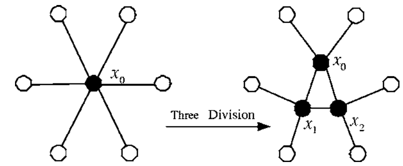

FIG. 1. Sketch maps for a three-division process on x_0. The solid circle in the left side is the node x_0 with degree 6. After a three-division process, x_0 is divided into three nodes x_0, x_1, and x_2 that are fully connected. The six edges incident from x_0 redistribute over these three nodes.

III. STRUCTURAL PERTURBATIONS

As mentioned above, nodes with very large betweenness, namely, hubs, may reduce the network synchronizability. So the present method of structural perturbations aims at these hubs. For a hub x_0, we add $m-1$ assistant nodes around it, labeled by $x_1, x_2, \ldots, x_{m-1}$. These m nodes are fully connected. Then, all the edges incident from x_0 will relink to a random picked node x_i $(i = 0, 1, \ldots, m-1)$. After this process, the betweenness of x_0 is divided into m almost equal parts associating with those m nodes. We call this process m division for short. A sketch map of a three-division process on node x_0 is shown in Fig. 1.

Due to the huge size of many real-life networks, it is usually impossible to obtain the nodes' betweenness. Fortunately, previous studies showed that there exists a strongly positive correlation between degree and betweenness in BA networks and some other real heterogeneity networks [27,28], that is to say, the node with larger degree will statistically have higher betweenness. Therefore, for practical reasons, we assume the node with higher betweenness is surely of larger degree in BA networks. So hereinafter, all the judgments and operations are based on the degree of nodes.

In order to enhance the network synchronizability, a few nodes with highest degree will be divided. Rank each node of a given network G according to its degree; the node that has highest degree is arranged at the top of the queue. Then, the network $G(\rho, m)$ can be obtained by the following $N\rho$ steps. First, carry out m division on the top node in G, leading to the network $G(1/N, m)$. Second, calculate all nodes' degree in $G(1/N, m)$, and rank each node according to its degree. Then, get the network $G(2/N, m)$ by dividing the top node in $G(1/N, m)$. Repeat this process $N\rho$ times; when $N\rho$ nodes have been divided in total, one will reach the network $G(\rho, m)$. Since randomness is involved in the dividing process, $G(\rho, m)$ is not unique. In this report, we focus on the difference between $G(\rho, m)$ and G.

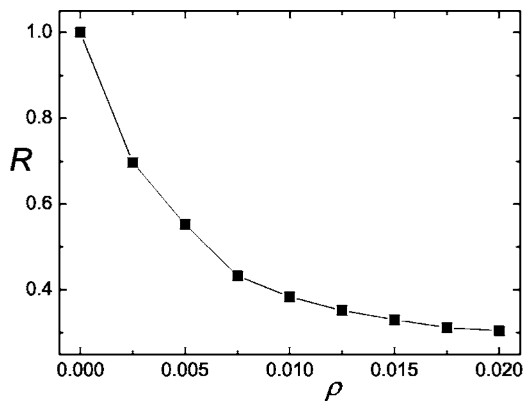

FIG. 2. Behavior of the ratio of the eigenratio of network $G(N\rho,m)$ to that of network G versus the number of divided nodes. As the number increases, the ratio is shown to be reduced, leading to better synchronization. The average is taken over 50 different network realizations.

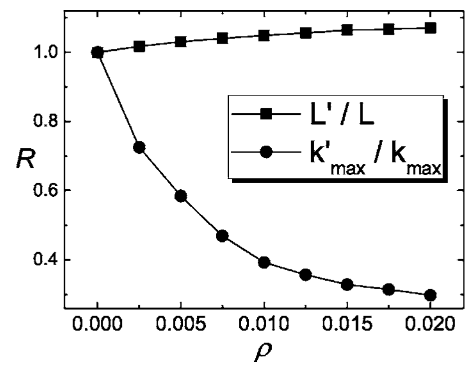

FIG. 3. The average distance L' and maximal degree $k'_{\max}$ in $G(\rho,m)$ vs ρ. L and $k_{\max}$ denote the average distance and maximal degree in the original network G. We plot the relative changes L'/L and $k'_{\max}/k_{\max}$ using squares and circles, respectively. One can see clearly that the dividing processes reduce the maximal degree while increasing the average distance. All the data are obtained by an average over 20 independent runs.

IV. SIMULATIONS

To explore how the structural perturbations affect network synchronizability, we compare the eigenratio r before and after the dividing processes. BA networks of size $N=2000$ and average degree $\langle k\rangle=12$ are used for simulations. In Fig. 2, we report the ratio $R=r'/r$ against the number of nodes that are divided, where r is the eigenratio of the original network and r' after the operation. Here we set $m=3$. With the probability ρ or the number of divided nodes increasing, the ratio R is observed to decrease, indicating the enhancement of synchronizability. In Fig. 2, it can be seen that to divide a very few nodes will sharply enhance the network synchronizability. R decreases to 0.7 when only five nodes are divided, and will drop to half after 0.6% nodes (i.e., 12 nodes) are divided.

To better understand the underlying mechanism of synchronization and the reason why these structural perturbations will greatly enhance network synchronizability, we investigate the behaviors of two extensively studied quantities, the average distance L and maximal degree $k_{\max}$. In BA networks, the node with maximal degree is most probably the very node having maximal betweenness. As illustrated in Fig. 3, L will increase with ρ, while $k_{\max}$ will decrease. This result provides some evidence of how the two factors affect the synchronization of systems. The maximal degree (i.e., the maximal betweenness) may play the main role in determining network synchronizability. It is worthwhile to emphasize that from the simulation results, we cannot say anything about how the average distance affects the network synchronizability. L varies slightly, and probably has nonsignificant influence compared with the change of $k_{\max}$. These results suggest that reducing the maximal betweenness of networks is a practical and effective approach to enhance the network synchronizability.

V. CONCLUSION AND DISCUSSION

Motivated by practical requirements and theoretical interest, numbers of scientists have begun to study how to enhance the network synchronizability, especially for scale-free networks [29,30]. These methods keep the network topology unchanged, while adding some weight into the system; thus the coupling matrix is changed. These approaches do not need any new nodes, new edges, or rewiring, but highly enhance the network synchronizability. In this Brief Report, we propose an approach to enhance the network synchronizability. This approach does not require any intelligence of nodes, but the network structure will be slightly changed. In some real-life communication networks such as the Internet, a long length edge may cost much more than a node or a short length edge [31,32], so if all the nodes $x_1, x_2, \cdots, x_{m-1}$ are in x_0 vicinal locations, our method is feasible.

Some recent work about network traffic dynamics reveals that the communication ability of the network, called the network throughput [5,6], is mainly determined by the maximal betweenness, thus to steer clear of those hub nodes may enhance the network throughput [6,33]. This is just the case of network synchronization. Some methods that can enhance the network throughput will enhance the network synchronizability too [5,29,30,33]. Therefore, we guess there may exist some common features between network traffic and network synchronization, although they seem completely irrelevant. We believe our work will enlighten readers on this subject, and is also relevant to traffic control on networks.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002); M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[2] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001); G. Yan, T. Zhou, J. Wang, Z.-Q. Fu, and B.-H. Wang, Chin. Phys. Lett. **22**, 510 (2005); T. Zhou, G. Yan, and B.-H. Wang, Phys. Rev. E **71**, 046141 (2005).
[3] A. E. Motter and Y.-C. Lai, Phys. Rev. E **66**, 065102(R) (2002); K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, Phys. Rev. Lett. **91**, 148701 (2003); T. Zhou and B. -H. Wang, Chin. Phys. Lett. **22**, 1072 (2005).
[4] B. Tadić, S. Thurner, and G. J. Rodgers, Phys. Rev. E **69**, 036102 (2004).
[5] L. Zhao, Y.-C. Lai, K. Park, and N. Ye, Phys. Rev. E **71**, 026125 (2005).
[6] G. Yan, T. Zhou, B. Hu, Z.-Q. Fu, and B.-H. Wang, e-print: cond-mat/0505366.
[7] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440(1998).
[8] C.-P. Zhu, S.-J. Xiong, Y.-J. Tian, N. Li, and K.-S. Jiang, Phys. Rev. Lett. **92**, 218702 (2004).
[9] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[10] In a network, the distance between two nodes is defined as the number of edges along the shortest path connecting them. The average distance L of the network, then, is defined as the mean distance between two nodes, averaged over all pairs of nodes.
[11] The clustering coefficient C denotes the probability that randomly picked two neighbors of a random selected node are neighbors.
[12] S. H. Strogatz and I. Stewart, Sci. Am. **269**, 102 (1993); C. M. Gray, J. Comput. Neurosci. **1**, 11(1994); L. Glass, Nature **410**, 277 (2001); Z. Néda, E. Ravasz, T. Vicsek, Y. Brechet, and A. L. Barabási, Phys. Rev. E **61**, 6987 (2000).
[13] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994); C. W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995); J. Jost and M. P. Joy, Phys. Rev. E **65**, 016201 (2001).
[14] P. M. Gade, Phys. Rev. E **54**, 64 (1996); S. C. Manrubia and A. S. Mikhailov, *ibid.* **60**, 1579 (1999).
[15] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000).
[16] P. M. Gade and C.-K. Hu, Phys. Rev. E **62**, 6409 (2000).
[17] X. F. Wang and G. Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 187 (2002).
[18] P.-Q. Jiang, B.-H. Wang, S.-L. Bu, Q.-H. Xia, and X.-S. Luo. Int. J. Mod. Phys. B **18**, 2674 (2004).
[19] M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[20] P. G. Lind, J. A. C. Gallas, and H. J. Herrmann, Phys. Rev. E **70**, 056207 (2004).
[21] The betweenness of a given node denotes the probability that a shortest path with random source and destination will pass through this node. See Ref. [22] for details.
[22] M. E. J. Newman, Phys. Rev. E **64**, 016131 (2001).
[23] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[24] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E **69**, 067105 (2004).
[25] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[26] G. Hu, J. Yang, and W. Liu, Phys. Rev. E **58**, 4440 (1998).
[27] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[28] M. Barthélemy, Eur. Phys. J. B **38**, 163 (2004).
[29] A. E. Motter, C. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005).
[30] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005).
[31] B. Waxman, IEEE J. Sel. Areas Commun. **6**, 1617 (1998).
[32] S.-H. Yook, H. Jeong, and A.-L. Barabási, Proc. Natl. Acad. Sci. U.S.A. **99**, 13382 (2002).
[33] C.-Y. Yin, B.-H. Wang, W.-X. Wang, T. Zhou, and H.-J. Yang, e-print: physics/0506204.